

# Soft-Collinear Messengers: A New Mode in Soft-Collinear Effective Theory

THOMAS BECHER<sup>(a)</sup>, RICHARD J. HILL<sup>(a)</sup> AND MATTHIAS NEUBERT<sup>(b)</sup>

<sup>(a)</sup>*Stanford Linear Accelerator Center, Stanford University  
Stanford, CA 94309, U.S.A.*

<sup>(b)</sup>*Newman Laboratory for Elementary-Particle Physics, Cornell University  
Ithaca, NY 14853, U.S.A.*

## Abstract

It is argued that soft-collinear effective theory for processes involving both soft and collinear partons, such as exclusive  $B$ -meson decays, should include a new mode in addition to soft and collinear ones. These “soft-collinear messengers” can interact with both soft and collinear particles without taking them far off-shell. They thus can communicate between the soft and collinear sectors of the theory. The relevance of the new mode is demonstrated with an explicit example, and the formalism incorporating the corresponding quark and gluon fields into the effective Lagrangian is developed.

# 1 Introduction

There is currently much effort devoted to applications of soft-collinear effective theory (SCET) [1, 2, 3, 4, 5] to exclusive  $B$  decays [6, 7, 8, 9, 10]. SCET provides a systematic framework in which to discuss QCD factorization theorems for these processes [11] and power corrections using the language of effective field theory. The hadronic states relevant to exclusive decays such as  $B \rightarrow K^* \gamma$  or  $B \rightarrow \pi \pi$  contain highly energetic, collinear partons inside the final-state light mesons, and soft partons inside the initial  $B$  meson. Understanding the intricate interplay between soft and collinear degrees of freedom is a challenge that one hopes to address using the effective theory. This interplay is relevant even in simpler processes such as semileptonic  $B$  decays near  $q^2 \approx 0$ , which are described in terms of heavy-to-light form factors at large recoil.

Power counting in SCET is based on an expansion parameter  $\lambda \sim \Lambda/E$ , where  $E \gg \Lambda_{\text{QCD}}$  is a large scale (typically  $E \sim m_b$  in  $B$  decays), and  $\Lambda \sim \Lambda_{\text{QCD}}$  is of order the QCD scale. A complication in SCET is that different components of particle momenta and fields may scale differently with the large scale  $E$ . To make this scaling explicit one introduces two light-like vectors  $n^\mu$  and  $\bar{n}^\mu$  satisfying  $n^2 = \bar{n}^2 = 0$  and  $n \cdot \bar{n} = 2$ . Typically,  $n^\mu$  is the direction of an outgoing fast hadron (or a jet of hadrons). Any 4-vector can then be decomposed as

$$p^\mu = (n \cdot p) \frac{\bar{n}^\mu}{2} + (\bar{n} \cdot p) \frac{n^\mu}{2} + p_\perp^\mu \equiv p_+^\mu + p_-^\mu + p_\perp^\mu, \quad (1)$$

where  $p_\perp \cdot n = p_\perp \cdot \bar{n} = 0$ . The light-like vectors  $p_\pm^\mu$  are defined by this relation. The relevant SCET degrees of freedom describing the partons in the external hadronic states of exclusive  $B$  decays are soft and collinear, where  $p_s^\mu \sim E(\lambda, \lambda, \lambda)$  for soft momenta and  $p_c^\mu \sim E(\lambda^2, 1, \lambda)$  for collinear momenta. Here and below we indicate the scaling properties of the components  $(n \cdot p, \bar{n} \cdot p, p_\perp)$ . The corresponding effective-theory fields and their scaling relations are  $h_v \sim \lambda^{3/2}$  (soft heavy quark),  $q_s \sim \lambda^{3/2}$  (soft light quark),  $A_s^\mu \sim (\lambda, \lambda, \lambda)$  (soft gluon), and  $\xi \sim \lambda$  (collinear quark),  $A_c^\mu \sim (\lambda^2, 1, \lambda)$  (collinear gluon). The collinear quark is described by a 2-component spinor subject to the constraint  $\not{n} \xi = 0$ .

Short-distance fluctuations in exclusive  $B$  decays are usually characterized by two different large scales: the hard scale  $E^2 \sim m_b^2$  associated with off-shell fluctuations of the heavy quark, and the “hard-collinear” scale  $p_s \cdot p_c \sim E\Lambda$  arising in interactions involving both soft and collinear degrees of freedom. In order to disentangle the physics associated with these two scales it is sometimes useful to perform the matching of full QCD onto the low-energy effective theory in two steps, by going through an intermediate effective theory (called SCET<sub>I</sub>) containing “hard-collinear” modes with virtualities  $p_{hc}^2 \sim E\Lambda$  as dynamical degrees of freedom. In a second step SCET<sub>I</sub> is matched onto the final theory (called SCET<sub>II</sub>) containing near on-shell soft and collinear partons only. In this paper we are concerned with the structure of this final effective theory.

At leading order in power counting the effective strong-interaction Lagrangian of SCET<sub>II</sub> splits up into separate Lagrangians for the soft and collinear fields. This property implies factorization of many processes involving soft and collinear partons at leading power in  $\lambda$ . Factorization is however not guaranteed for quantities that vanish at leading power, such

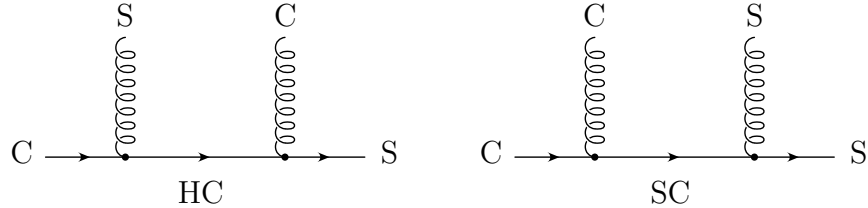


Figure 1: Examples of interactions between two soft and two collinear fields induced by the exchange of a hard-collinear particle (left) and of a soft-collinear particle (right). Hard-collinear modes are integrated out in SCET<sub>II</sub>, while soft-collinear modes remain as low-energy degrees of freedom.

as heavy-to-light form factors at large recoil. It was argued in [5] that at subleading order in  $\lambda$  interactions between soft and collinear particles occur, which can violate factorization. As illustrated in Figure 1, these interactions can be mediated by the exchange of a short-distance hard-collinear mode, or by a long-distance “messenger particle” with momentum scaling  $E(\lambda^2, \lambda, \dots)$ , where the transverse momentum components can be at most of  $O(\lambda)$ . We will argue in the present work that these two exchange mechanisms contribute under different kinematic conditions. In particular, the hard-collinear exchange requires the presence of collinear particles in the initial state and so is irrelevant for SCET applications to  $B$  decays. The scaling of the long-distance messenger particle is such that it can couple to both soft and collinear fields without taking them far off-shell. It was left open in [5] whether the messenger exchange should be described in terms of a new field in the effective theory, or by considering the exchange particle as a soft (or collinear) field subject to certain constraints on some of its momentum components.

To answer this question, one must determine whether propagators with scaling corresponding to the messenger particles can give rise to pinch singularities in Feynman loop diagrams. By the Coleman–Norton theorem such singularities only arise from on-shell intermediate states [12], and hence the new mode would have to have momentum scaling  $p_{sc}^\mu \sim E(\lambda^2, \lambda, \lambda^{3/2})$ . Since this is the “largest” on-shell mode that can couple to both soft and collinear fields without affecting their momentum scaling, we call this mode “soft-collinear”. Naively, one would expect that the low-energy effective theory would only contain soft and collinear fields scaling in the same way as the external momenta of soft and collinear hadrons, especially since the soft-collinear momentum corresponds to a virtuality  $p_{sc}^2 \sim E^2\lambda^3$  that, for  $\lambda \sim \Lambda/E$ , is below the scale  $\Lambda^2$ . However, the evaluation of sample one-loop diagrams reveals that the situation is more complicated. The interplay of soft and collinear kinematics makes it necessary to introduce modes with virtuality  $E^2\lambda$  (hard-collinear) and  $E^2\lambda^3$  (soft-collinear) in addition to collinear and soft modes. In the low-energy theory the modes with off-shellness  $E^2\lambda$  are integrated out and lead to the occurrence of operators which are smeared over large distances of  $O(1/\Lambda)$  [5], while the soft-collinear modes have to be kept as degrees of freedom.

The first goal of this paper is to establish, with the help of an explicit example, that soft-collinear modes are part of the low-energy effective theory for soft and collinear partons. As a result, SCET<sub>II</sub> is a more complicated theory than anticipated, and the matching of the intermediate effective theory SCET<sub>I</sub> onto the final theory SCET<sub>II</sub> is more involved than

envisioned in [8, 10, 13]. In the second part of the paper we develop SCET<sub>II</sub> in the presence of the new modes and discuss some examples of the relevance of soft-collinear messenger exchange.

A prominent example of quantities that are sensitive to soft-collinear exchange graphs are heavy-to-light form factors at large recoil, for which soft-collinear modes can (and do) contribute at first order in  $\lambda$ . They are needed to describe the “soft overlap” contribution, which is formally the leading contribution to the form factors in the heavy-quark limit [14]. Another important application of the soft-collinear modes arises when one studies the endpoint behavior of hard-scattering kernels in QCD factorization theorems. The demonstration of the absence of endpoint singularities is an important part of factorization proofs (see, e.g., the discussion in [9]). In the endpoint region  $x \ll 1$  the scaling of the momentum of a collinear parton carrying longitudinal momentum fraction  $x$  inside a fast light hadron changes from  $E(\lambda^2, 1, \lambda)$  to  $E(\lambda^2, \lambda, \dots)$ . Similarly, in the region  $l_+ \ll \Lambda$  the scaling of the momentum of a soft parton inside the  $B$  meson changes from  $E(\lambda, \lambda, \lambda)$  to  $E(\lambda^2, \lambda, \dots)$ . In both cases it is natural to describe these endpoint configurations in terms of soft-collinear fields. For the case of factorization for the exclusive decay  $B \rightarrow K^* \gamma$  this will be illustrated in [15].

## 2 Relevance of the soft-collinear mode

It will be instructive to demonstrate the relevance of soft-collinear modes with an explicit example. Consider the scalar triangle graph shown in Figure 2 in the kinematic region where the external momenta are  $l^\mu \sim (\lambda, \lambda, \lambda)$  soft,  $p^\mu \sim (\lambda^2, 1, \lambda)$  collinear, and  $q^\mu = (l - p)^\mu \sim (\lambda, 1, \lambda)$  hard-collinear. We define the loop integral

$$I = i\pi^{-d/2} \mu^{4-d} \int d^d k \frac{1}{(k^2 + i0) [(k+l)^2 + i0] [(k+p)^2 + i0]} \quad (2)$$

in  $d = 4 - 2\epsilon$  space-time dimensions and analyze it for arbitrary external momenta obeying the above scaling relations. It will be convenient to define the invariants

$$L^2 \equiv -l^2 - i0, \quad P^2 \equiv -p^2 - i0, \quad Q^2 \equiv -(l-p)^2 - i0 = 2l_+ \cdot p_- - i0 + \dots, \quad (3)$$

which scale like  $L^2, P^2 \sim \lambda^2$  and  $Q^2 \sim \lambda$ . (In physical units,  $L^2, P^2 \sim \Lambda^2$  and  $Q^2 \sim E\Lambda$  with  $E \gg \Lambda$ .) As long as these momenta are off-shell the integral is ultra-violet and infra-red finite and can be evaluated setting  $\epsilon = 0$ , with the result

$$I = \frac{1}{Q^2} \left[ \ln \frac{Q^2}{L^2} \ln \frac{Q^2}{P^2} + \frac{\pi^2}{3} + O(\lambda) \right]. \quad (4)$$

Let us now try to reproduce this result by evaluating the contributions from different momentum modes. The method of regions [16, 17] can be used to find the momentum configurations giving rise to leading-order contributions to the integral  $I$ . There is no hard contribution, since for  $k^\mu \sim E(1, 1, 1)$  the integrand can be Taylor-expanded in the external momenta, giving scaleless integrals that vanish in dimensional regularization. A short-distance contribution

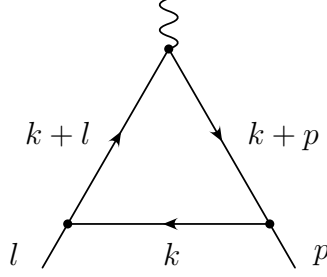


Figure 2: Scalar triangle diagram with an external soft momentum  $l$  and a collinear momentum  $p$ . The loop momentum is denoted by  $k$ .

arises from the region of hard-collinear loop momenta  $k^\mu \sim E(\lambda, 1, \lambda^{1/2})$ , and power counting shows that it is indeed of leading power:  $I_{\text{HC}} \sim \lambda^2 \cdot (\lambda^{-1})^3 \sim \lambda^{-1}$  (where we display the scaling of the integration measure and of the three propagators), which is of the same order as the leading term in the result (4). Simplifying the propagators in the hard-collinear region we obtain at leading power

$$\begin{aligned}
I_{\text{HC}} &= i\pi^{-d/2} \mu^{4-d} \int d^d k \frac{1}{(k^2 + i0)(k^2 + 2k_- \cdot l_+ + i0)(k^2 + 2k_+ \cdot p_- + i0)} \\
&= \frac{\Gamma(1+\epsilon)}{2l_+ \cdot p_-} \frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} \left( \frac{\mu^2}{2l_+ \cdot p_-} \right)^\epsilon \\
&= \frac{\Gamma(1+\epsilon)}{Q^2} \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2}{Q^2} + \frac{1}{2} \ln^2 \frac{\mu^2}{Q^2} - \frac{\pi^2}{6} \right) + O(\epsilon),
\end{aligned} \tag{5}$$

where in the last step we have replaced  $2l_+ \cdot p_- \rightarrow Q^2$ , which is legitimate at leading power. The relevant physical scale of this contribution is the hard-collinear scale  $\mu^2 \sim Q^2 \sim E\Lambda$ .

Long-distance contributions to the integral arise from the regions of soft or collinear loop momenta, where  $k^\mu \sim E(\lambda, \lambda, \lambda)$  or  $k^\mu \sim E(\lambda^2, 1, \lambda)$ , respectively. Power counting shows that both regions give rise to leading-order contributions:  $I_{\text{S}} \sim \lambda^4 \cdot (\lambda^{-2})^2 \cdot \lambda^{-1} \sim \lambda^{-1}$ , and  $I_{\text{C}} \sim \lambda^4 \cdot \lambda^{-2} \cdot \lambda^{-1} \cdot \lambda^{-2} \sim \lambda^{-1}$ . Simplifying the propagators in the soft region we obtain at leading power

$$\begin{aligned}
I_{\text{S}} &= i\pi^{-d/2} \mu^{4-d} \int d^d k \frac{1}{(k^2 + i0)[(k+l)^2 + i0](2k_+ \cdot p_- + i0)} \\
&= -\frac{\Gamma(1+\epsilon)}{2l_+ \cdot p_-} \frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} \left( \frac{\mu^2}{L^2} \right)^\epsilon \\
&= \frac{\Gamma(1+\epsilon)}{Q^2} \left( -\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\mu^2}{L^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{L^2} + \frac{\pi^2}{6} \right) + O(\epsilon).
\end{aligned} \tag{6}$$

The relevant physical scale of this contribution is the soft scale  $\mu^2 \sim L^2 \sim \Lambda^2$ . Similarly, in

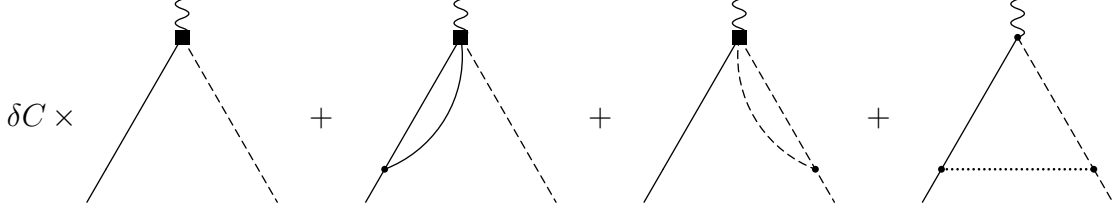


Figure 3: Effective field-theory graphs. Full lines denote soft fields, dashed lines collinear fields, and dotted lines soft-collinear fields.

the collinear region we obtain

$$\begin{aligned}
I_C &= i\pi^{-d/2} \mu^{4-d} \int d^d k \frac{1}{(k^2 + i0) (2k_- \cdot l_+ + i0) [(k+p)^2 + i0]} \\
&= -\frac{\Gamma(1+\epsilon)}{2l_+ \cdot p_-} \frac{\Gamma^2(-\epsilon)}{\Gamma(1-2\epsilon)} \left( \frac{\mu^2}{P^2} \right)^\epsilon \\
&= \frac{\Gamma(1+\epsilon)}{Q^2} \left( -\frac{1}{\epsilon^2} - \frac{1}{\epsilon} \ln \frac{\mu^2}{P^2} - \frac{1}{2} \ln^2 \frac{\mu^2}{P^2} + \frac{\pi^2}{6} \right) + O(\epsilon).
\end{aligned} \tag{7}$$

The relevant physical scale of this contribution is the collinear scale  $\mu^2 \sim P^2 \sim \Lambda^2$ .

Figure 3 illustrates that the three contributions derived above have a representation in terms of a low-energy effective theory containing soft and collinear fields, in which the hard-collinear modes are integrated out. Here  $\delta C$  denotes the hard-collinear contribution to the Wilson coefficient of the current operator containing a soft and a collinear field. This coefficient arises from integrating out the short-distance hard-collinear modes. In the second and third diagrams the hard-collinear propagators have been shrunk to a point, leaving loops of only soft or only collinear lines.

The sum  $I_{\text{HC}} + I_C + I_S$  does not reproduce the exact leading-order term in (4). In fact, the two expressions differ by large single and double (Sudakov) logarithms of the form  $\ln(\mu^2 Q^2 / L^2 P^2)$ , which remain large even at a low scale  $\mu^2 \sim \Lambda^2$ . The discrepancy is due to the presence of another leading region, which arises when the loop momentum scales like  $k^\mu \sim E(\lambda^2, \lambda, \lambda^{3/2})$ . Power counting shows that this indeed gives rise to a leading-order contribution:  $I_{\text{SC}} \sim \lambda^6 \cdot \lambda^{-3} \cdot (\lambda^{-2})^2 \sim \lambda^{-1}$ . Simplifying the propagators in the soft-collinear region we find

$$\begin{aligned}
I_{\text{SC}} &= i\pi^{-d/2} \mu^{4-d} \int d^d k \frac{1}{(k^2 + i0) (2k_- \cdot l_+ + l^2 + i0) (2k_+ \cdot p_- + p^2 + i0)} \\
&= -\frac{\Gamma(1+\epsilon)}{2l_+ \cdot p_-} \Gamma(\epsilon) \Gamma(-\epsilon) \left( \frac{2\mu^2 l_+ \cdot p_-}{L^2 P^2} \right)^\epsilon \\
&= \frac{\Gamma(1+\epsilon)}{Q^2} \left( \frac{1}{\epsilon^2} + \frac{1}{\epsilon} \ln \frac{\mu^2 Q^2}{L^2 P^2} + \frac{1}{2} \ln^2 \frac{\mu^2 Q^2}{L^2 P^2} + \frac{\pi^2}{6} \right) + O(\epsilon).
\end{aligned} \tag{8}$$

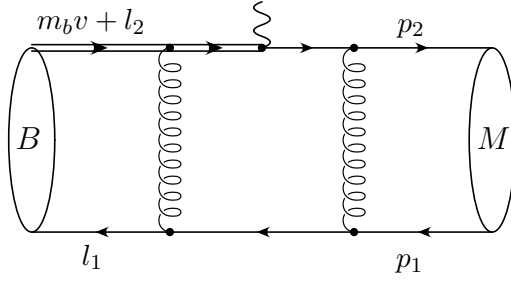


Figure 4: A QCD diagram contributing to the decay of a  $B$  meson to an energetic light meson  $M$ . The relevant loop subgraph is a pentagon with two collinear external lines with momenta  $p_1$ ,  $p_2$ , a soft line with momentum  $l_1$ , and a heavy-quark line with momentum  $m_b v + l_2$ .

The relevant scale of this contribution is a particular combination of the hard-collinear, soft, and collinear scales,  $\mu^2 \sim (L^2 P^2)/Q^2 \sim \Lambda^3/E$ , which (for  $\lambda \sim \Lambda/E$ ) is parametrically smaller than the QCD scale  $\Lambda^2$ .

The soft-collinear contribution precisely accounts for the difference encountered above, so that

$$I = I_{\text{HC}} + I_{\text{S}} + I_{\text{C}} + I_{\text{SC}} \quad (9)$$

up to higher-order terms in  $\lambda$ . This example demonstrates that soft-collinear modes are required to reproduce the correct analytic structure of full-theory amplitudes containing both soft and collinear external momenta. One must then introduce a new field in the low-energy effective theory, whose contribution is represented by the last diagram in Figure 3.

The above conclusion is completely general and does not rely on the particular diagram investigated here. For instance, we have analyzed the corresponding vertex diagram in QCD, as well as the QCD pentagon subgraph shown in Figure 4, which contributes to heavy-to-light form factors at large recoil. We again find that soft-collinear modes are necessary to reproduce the correct analytic structure of the diagrams, i.e., the QCD vertex graph receives a leading-order contribution from the soft-collinear exchange graph shown in the last diagram in Figure 3, and the pentagon subgraph in Figure 4 receives a leading-order contribution from the region where the anti-quark line between the two lower gluon attachments carries a soft-collinear momentum.

### 3 Relation with the Sudakov form factor

In order to build up intuition for the new soft-collinear mode it may be instructive to consider the following analogy with the off-shell Sudakov form factor. Starting from the kinematic situation in Figure 2 we can perform a longitudinal Lorentz boost into the Breit frame, in which the two 3-momenta  $\vec{l}$  and  $\vec{p}$  are equal in magnitude and opposite in direction. This boost rescales the components  $(n \cdot q, \bar{n} \cdot q, q_\perp)$  of all 4-momenta into  $(\lambda^{-1/2} n \cdot q, \lambda^{1/2} \bar{n} \cdot q, q_\perp)$ .

Introducing a new expansion parameter  $\hat{\lambda} \equiv \lambda^{1/2}$  we then find the following correspondence between modes in the original frame and in the Breit frame:

$$\begin{array}{llll}
\text{hard-collinear:} & E(\lambda, 1, \lambda^{1/2}) & \leftrightarrow & \text{hard:} & \hat{E}(1, 1, 1) \\
\text{soft:} & E(\lambda, \lambda, \lambda) & \leftrightarrow & \text{anti-collinear:} & \hat{E}(1, \hat{\lambda}^2, \hat{\lambda}) \\
\text{collinear:} & E(\lambda^2, 1, \lambda) & \leftrightarrow & \text{collinear:} & \hat{E}(\hat{\lambda}^2, 1, \hat{\lambda}) \\
\text{soft-collinear:} & E(\lambda^2, \lambda, \lambda^{3/2}) & \leftrightarrow & \text{ultra-soft:} & \hat{E}(\hat{\lambda}^2, \hat{\lambda}^2, \hat{\lambda}^2)
\end{array} \tag{10}$$

where  $\hat{E} \equiv E\hat{\lambda}$ . These are precisely the modes arising in the analysis of the off-shell Sudakov form factor [18, 19]. In the language of effective field theory, it follows that the original SCET<sub>II</sub> problem (with expansion parameter  $\lambda$  and large scale  $E$ ) can be mapped onto a SCET<sub>I</sub> problem (with expansion parameter  $\hat{\lambda}$  and large scale  $\hat{E}$ ) containing two types of collinear fields along with ultra-soft fields, which correspond to the soft-collinear messenger modes of the original problem.<sup>1</sup> We hope this analogy with a familiar problem will help to convince the reader of the relevance of soft-collinear modes in SCET<sub>II</sub>. We now proceed to construct the low-energy effective theory including the corresponding fields.

## 4 The soft-collinear Lagrangian

We start by studying the scaling properties and self-interactions of soft-collinear fields. We introduce soft-collinear gauge and fermion fields,  $A_{sc}^\mu$  and  $q_{sc}$ , in the usual way. The scaling properties of these fields follow from an analysis of the corresponding two-point functions in position space [4], taking into account that  $p_{sc}^2 \sim \lambda^3$  and  $d^4p_{sc} \sim \lambda^6$ . We find that

$$A_{sc}^\mu \sim (\lambda^2, \lambda, \lambda^{3/2}) \tag{11}$$

scales like a soft-collinear momentum, which guarantees homogeneous scaling laws for the components of the soft-collinear covariant derivative  $iD_{sc}^\mu = i\partial^\mu + A_{sc}^\mu$ . Here and below, a factor of  $g_s$  is included in the definition of the gauge fields.

The fermion field can be split up into large and small components with different scaling relations. We define  $q_{sc} = \theta + \sigma$ , where

$$\theta = \frac{\not{n}\not{\bar{n}}}{4} q_{sc}, \quad \sigma = \frac{\not{\bar{n}}\not{n}}{4} q_{sc}, \quad \text{with} \quad \not{n}\theta = \not{\bar{n}}\sigma = 0. \tag{12}$$

The analysis of the fermion two-point function reveals that

$$\theta \sim \lambda^2, \quad \sigma \sim \lambda^{5/2}. \tag{13}$$

As long as we consider interactions of only soft-collinear fields, nothing prevents us from boosting to a Lorentz frame in which these fields have homogeneous momentum scaling  $p^\mu \sim$

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<sup>1</sup>However, this mapping cannot be done for processes involving heavy quarks, in which a natural Lorentz frame is defined by the rest frame of a heavy hadron.



$E(\lambda^{3/2}, \lambda^{3/2}, \lambda^{3/2})$ . This is analogous to the case of the collinear Lagrangian, which in this sense is equivalent to the ordinary QCD Lagrangian. It follows that the effective Lagrangian for soft-collinear fields has the same form as the collinear Lagrangian [1, 4], i.e.

$$\mathcal{L}_{sc} = \bar{\theta} \frac{\not{n}}{2} i n \cdot D_{sc} \theta - \bar{\theta} i \not{D}_{sc\perp} \frac{1}{2} \frac{1}{i \bar{n} \cdot D_{sc}} i \not{D}_{sc\perp} \theta. \quad (14)$$

To obtain this result one simply inserts the decomposition  $q_{sc} = \theta + \sigma$  into the Dirac Lagrangian and eliminates the small-component field  $\sigma$  using its equation of motion, which yields

$$\sigma = -\frac{\not{n}}{2} \frac{1}{i \bar{n} \cdot D_{sc}} i \not{D}_{sc\perp} \theta. \quad (15)$$

It is straightforward to check that the operators in the effective Lagrangian (14) scale like  $\lambda^6$ , which when combined with the scaling of the soft-collinear measure  $d^4x \sim \lambda^{-6}$  ensures that these terms are of leading order in power counting. The fermion Lagrangian given above must be complemented by the pure-gauge and ghost Lagrangians, which retain the same form as in ordinary QCD. Using arguments along the lines of [4] it can be shown that the Lagrangian  $\mathcal{L}_{sc}$  is not renormalized. Below, we will often write expressions in terms of the two components  $\theta$  and  $\sigma$ , keeping in mind that at the end  $\sigma$  may be eliminated using (15).

## 5 Interactions of soft-collinear fields with other fields

The effective Lagrangian of SCET<sub>II</sub> can be derived by decomposing the QCD fields into the various modes and integrating out the hard and hard-collinear modes. Here we focus on the pure QCD Lagrangian without external operators mediating weak interactions. In general, the effective Lagrangian can be split up as

$$\mathcal{L}_{\text{SCET}_{\text{II}}} = \mathcal{L}_s + \mathcal{L}_c + \mathcal{L}_{sc} + \mathcal{L}_{s+sc}^{\text{int}} + \mathcal{L}_{c+sc}^{\text{int}} \left[ + \mathcal{L}_{s+c}^{\text{int}} \right], \quad (16)$$

where the first three terms correspond to the Lagrangians of soft particles (including heavy quarks), collinear particles, and soft-collinear particles. The term  $\mathcal{L}_{s+c}^{\text{int}}$  in brackets corresponds to effective interactions among soft and collinear particles induced by the exchange of hard-collinear modes, which arise at subleading order in  $\lambda$  [5]. We will argue in Section 6 that these interactions are kinematically forbidden in  $B$  decays. This term can thus be dropped from the SCET Lagrangian. Our focus in this paper is on the terms  $\mathcal{L}_{s+sc}^{\text{int}}$  and  $\mathcal{L}_{c+sc}^{\text{int}}$ , describing the interactions involving soft-collinear fields. Note that the integration measures  $d^4x$  in the action  $S_{\text{SCET}_{\text{II}}} = \int d^4x \mathcal{L}_{\text{SCET}_{\text{II}}}$  scale differently for the various terms above. The measures for  $\mathcal{L}_s$  and  $\mathcal{L}_c$  scale like  $\lambda^{-4}$ , while that for  $\mathcal{L}_{sc}$  scales like  $\lambda^{-6}$ . The measure for the interaction Lagrangian  $\mathcal{L}_{s+c}^{\text{int}}$  scales like  $\lambda^{-3}$ , whereas those for the interaction Lagrangians  $\mathcal{L}_{s+sc}^{\text{int}}$  and  $\mathcal{L}_{c+sc}^{\text{int}}$  scale like  $\lambda^{-4}$  (see below).

Soft-collinear fields can couple to collinear or soft fields without altering their scaling properties. Momentum conservation implies that in QCD (i.e., at the level of three- and four-point vertices) soft-collinear fields can only couple to either soft or collinear modes, but not

both. In such interactions more than one soft or collinear particle must be involved. These “pure QCD” interactions are always near on-shell and do not involve the exchange of hard or hard-collinear modes, so that we can perform the construction of the effective theory at tree level. The three relevant regions are collinear, soft, and soft-collinear. We thus split up the gluon field as  $A^\mu = A_c^\mu + A_s^\mu + A_{sc}^\mu$  and choose a similar decomposition for the quark field. The QCD Lagrangian is then expanded in these fields, dropping terms that are forbidden by momentum conservation. Due to the above form of the Lagrangian we can separately construct the effective theory in the  $(s + sc)$  and  $(c + sc)$  sectors. The resulting effective interaction Lagrangians  $\mathcal{L}_{s+sc}^{\text{int}}$  and  $\mathcal{L}_{c+sc}^{\text{int}}$  are not renormalized.

An important remark is that the soft-collinear fields in interactions with soft or collinear fields must be multipole expanded in order to properly separate the contributions from the different momentum regions and to avoid double counting [4]. Consider a term in the action containing some collinear fields and some soft-collinear fields, e.g.

$$\int d^4x \phi_c(x) \phi_c(x) \phi_{sc}(x) \sim \int d^4x e^{i(p+p'+q)\cdot x}, \quad (17)$$

where  $p, p'$  are collinear momenta and  $q$  is a soft-collinear momentum. Integration over  $d^4x$  in the action gives rise to  $\delta$ -functions, which must be used to eliminate one of the collinear momenta, not the soft-collinear momentum. These  $\delta$ -functions scale like  $\lambda^{-4}$ , which is thus the scaling of the measure  $d^4x$ . The vector  $x^\mu$  scales as appropriate for the argument of a collinear field, i.e.,  $x^\mu \sim (1, \lambda^{-2}, \lambda^{-1})$ . The soft-collinear scaling  $q^\mu \sim (\lambda^2, \lambda, \lambda^{3/2})$  then implies that we can expand  $e^{iq\cdot x} = e^{iq_+\cdot x_-} (1 + iq_\perp \cdot x_\perp + \dots)$ . For the soft-collinear field this implies the multipole expansion

$$\phi_{sc}(x) = \phi_{sc}(x_-) + x_\perp \cdot \partial_\perp \phi_{sc}(x_-) + \dots \quad (\text{in collinear interactions}), \quad (18)$$

where  $x_-^\mu = \frac{1}{2}(\bar{n} \cdot x) n^\mu$ . The first correction term is of  $O(\lambda^{1/2})$ , and the omitted terms are of  $O(\lambda)$  and higher. Similarly, if the soft-collinear field interacts with soft fields the vector  $x^\mu$  scales as appropriate for a soft momentum, i.e.,  $x^\mu \sim (\lambda^{-1}, \lambda^{-1}, \lambda^{-1})$ , and so only the dependence of the soft-collinear field on  $x_+$  must be kept, i.e.

$$\phi_{sc}(x) = \phi_{sc}(x_+) + x_\perp \cdot \partial_\perp \phi_{sc}(x_+) + \dots \quad (\text{in soft interactions}), \quad (19)$$

where  $x_+^\mu = \frac{1}{2}(n \cdot x) \bar{n}^\mu$ .

Soft-collinear fields can couple to soft or collinear fields without altering their scaling properties. This motivates the treatment of the soft-collinear gluon field as a background field, which is smoother than soft and collinear fields [4]. It is also convenient to choose fermion field variables which do not mix under gauge transformations. Thus under soft gauge transformations  $U_s$  the soft fields transform as

$$A_s^\mu \rightarrow U_s A_s^\mu U_s^\dagger + U_s [iD_{sc}^\mu, U_s^\dagger], \quad q_s \rightarrow U_s q_s, \quad (20)$$

while the collinear and soft-collinear fields remain invariant. Likewise, under collinear gauge transformations  $U_c$  the collinear fields transform as

$$A_c^\mu \rightarrow U_c A_c^\mu U_c^\dagger + U_c [iD_{sc}^\mu, U_c^\dagger], \quad \xi \rightarrow U_c \xi, \quad (21)$$

while the soft and soft-collinear fields remain invariant. Finally, under soft-collinear gauge transformations  $U_{sc}$  the fields transform as

$$\begin{aligned}
A_c^\mu &\rightarrow U_{sc} A_c^\mu U_{sc}^\dagger, & \xi &\rightarrow U_{sc} \xi, \\
A_s^\mu &\rightarrow U_{sc} A_s^\mu U_{sc}^\dagger, & q_s &\rightarrow U_{sc} q_s, \\
A_{sc}^\mu &\rightarrow U_{sc} A_{sc}^\mu U_{sc}^\dagger + U_{sc} [i\partial^\mu, U_{sc}^\dagger], & q_{sc} &\rightarrow U_{sc} q_{sc}.
\end{aligned} \tag{22}$$

It can be seen from these relations that the combination  $(A_s^\mu + A_{sc}^\mu)$  transforms in the usual way under both soft and soft-collinear gauge transformations, while  $(A_c^\mu + A_{sc}^\mu)$  transforms in the usual way under both collinear and soft-collinear gauge transformations. The sum of the corresponding fermion fields, however, do not transform as the QCD fermion field. Our strategy will therefore be to adopt a particular gauge in the soft and collinear sectors of the theory, restoring gauge invariance at a later stage. Specifically, we adopt soft light-cone gauge  $n \cdot A_s = 0$  (SLCG) and collinear light-cone gauge  $\bar{n} \cdot A_c = 0$  (CLCG).

Another subtlety related to the implementation of the gauge transformations (20)–(22) is that they change the scaling behavior of the various fields, because the components of the covariant derivative  $D_{sc}^\mu$  acting on soft or collinear fields do not have homogeneous scaling behavior, and also because the soft-collinear fields multiplying soft or collinear fields are not multipole expanded. In the following we discuss how to set up homogeneous gauge transformations that preserve the power counting of the fields, following closely the treatment in [20]. This discussion is necessarily rather technical. The reader not interested in the details of the derivation may directly consult the final results given in (38) and (49).

## 5.1 Interactions between soft-collinear fields and soft fields

We start with the sector of the theory involving soft and soft-collinear fields. In order to have a well-defined power counting, we replace the transformation rules for soft fields in (20) and (22) by the homogeneous gauge transformations

$$\begin{aligned}
\text{soft: } n \cdot A_s &\rightarrow U_s n \cdot A_s U_s^\dagger + U_s [in \cdot \partial, U_s^\dagger], & q_s &\rightarrow U_s q_s, \\
A_{s\perp}^\mu &\rightarrow U_s A_{s\perp}^\mu U_s^\dagger + U_s [i\partial_\perp^\mu, U_s^\dagger], \\
\bar{n} \cdot A_s &\rightarrow U_s \bar{n} \cdot A_s U_s^\dagger + U_s [i\bar{n} \cdot D_{sc}(x_+), U_s^\dagger], \\
\text{soft-collinear: } A_s^\mu &\rightarrow U_{sc}(x_+) A_s^\mu U_{sc}^\dagger(x_+), & q_s &\rightarrow U_{sc}(x_+) q_s,
\end{aligned} \tag{23}$$

which are obtained by consistently keeping the leading-order terms in each of the original transformation rules. The soft-collinear fields transform in the same way as before. Here and below we use the notation that fields without argument live at position  $x$ , whereas some of the soft-collinear fields live at position  $x_+$  as indicated.

The new soft quark and gluon fields obeying the homogeneous transformation rules are related to the original ones by a (field-dependent, non-linear, and non-local) field redefinition.

As shown in [20], soft fields  $\hat{q}_s$  and  $\hat{A}_s^\mu$  having these gauge transformations are given by

$$\begin{aligned} q_s|_{\text{SLCG}} &= R_s S_s^\dagger \hat{q}_s, \\ A_{s\perp}^\mu|_{\text{SLCG}} &= R_s S_s^\dagger (i\hat{D}_{s\perp}^\mu S_s) R_s^\dagger, \\ \bar{n} \cdot A_s|_{\text{SLCG}} &= R_s \left[ S_s^\dagger (i\bar{n} \cdot \hat{D}_s S_s) + S_s^\dagger \bar{n} \cdot A_{sc}(x_+) S_s - \bar{n} \cdot A_{sc}(x_+) \right] R_s^\dagger, \end{aligned} \quad (24)$$

where the fields on the left-hand side are in soft light-cone gauge. The quantity

$$S_s(x) \equiv \text{P exp} \left( i \int_{-\infty}^0 dt n \cdot \hat{A}_s(x + tn) \right) \quad (25)$$

with  $t \sim \lambda^{-1}$  is a soft Wilson line (expressed in terms of the new gluon field) along the  $n$ -direction, and

$$R_s(x) = \text{P exp} \left( i \int_0^1 dt (x - x_+)_\mu A_{sc}^\mu(x_+ + t(x - x_+)) \right) \quad (26)$$

is the gauge string of soft-collinear fields from  $x_+$  to  $x$ . This quantity differs from 1 by terms of  $O(\lambda^{1/2})$  and so can be expanded; this will be used below. Note that  $S_s$  has the simple transformation properties

$$S_s \rightarrow U_s S_s, \quad S_s \rightarrow U_{sc}(x_+) S_s U_{sc}^\dagger(x_+), \quad (27)$$

because the arguments of the soft fields in the path-ordered exponential correspond to the same  $x_+$ . The quantity  $R_s$  is invariant under soft gauge transformations and transforms like

$$R_s(x) \rightarrow U_{sc}(x) R_s(x) U_{sc}^\dagger(x_+) \quad (28)$$

under soft-collinear gauge transformations. It follows that the expressions on the right-hand side of (24) are invariant under soft gauge transformations and transform as ordinary QCD quark and gluon fields (at position  $x$ , not  $x_+$ ) under soft-collinear gauge transformations. The interpretation of (24) is that the gauge transformation  $S_s$  puts the hatted fields in soft light-cone gauge and the  $R_s$  transformation “de-homogenizes” them, i.e., it converts the fields with homogeneous transformation laws into fields satisfying ordinary gauge transformations.

To obtain the effective Lagrangian  $\mathcal{L}_{s+sc}^{\text{int}}$  in (16) we adopt soft light-cone gauge and insert the decomposition  $q = q_s + q_{sc}$  into the Dirac Lagrangian, dropping terms that are forbidden by momentum conservation. This yields

$$\bar{q} i \not{D} q|_{\text{SLCG}} \rightarrow \bar{q}_s i \not{D}_{s+sc} q_s + \bar{q}_s \not{A}_s q_{sc} + \bar{q}_{sc} \not{A}_s q_s + \bar{q}_{sc} i \not{D}_{sc} q_{sc}. \quad (29)$$

After elimination of the small-component field  $\sigma$  the last term gives rise to the soft-collinear Lagrangian discussed in Section 4. Let us then focus on the remaining terms and express them

in terms of the homogenized fields defined in (24). After a straightforward calculation we find that

$$\begin{aligned}
\mathcal{L}_{\text{quark}} \rightarrow & \bar{q}_s \left( i \hat{\mathcal{D}}_s + \frac{\not{n}}{2} \bar{n} \cdot A_{sc}(x_+) \right) \hat{q}_s + \bar{q}_s S_s \left( R_s^\dagger i \mathcal{D}_{sc} R_s - i \not{\partial} - \frac{\not{n}}{2} \bar{n} \cdot A_{sc}(x_+) \right) S_s^\dagger \hat{q}_s \\
& + \left\{ \bar{q}_s S_s \frac{\not{n}}{2} \left[ S_s^\dagger (i \bar{n} \cdot \hat{D}_s S_s) + S_s^\dagger \bar{n} \cdot A_{sc}(x_+) S_s - \bar{n} \cdot A_{sc}(x_+) \right] R_s^\dagger q_{sc} \right. \\
& \left. + \bar{q}_s (i \hat{\mathcal{D}}_{s\perp} S_s) R_s^\dagger q_{sc} + \text{h.c.} \right\}.
\end{aligned} \tag{30}$$

From now on we will drop the “hat” on the redefined soft fields.

To put this Lagrangian in a useful form we expand the various quantities involving  $R_s$  in powers of  $\lambda^{1/2}$  [20]. We need

$$\begin{aligned}
R_s^\dagger(x) q_{sc}(x) &= \theta(x_+) + \sigma(x_+) + x_\perp \cdot D_{sc}(x_+) \theta(x_+) + O(\lambda^3), \\
R_s^\dagger i \mathcal{D}_{sc} R_s - i \not{\partial} - \frac{\not{n}}{2} \bar{n} \cdot A_{sc}(x_+) &= \frac{\not{n}}{2} x_{\perp\mu} \bar{n}_\nu g_s G_{sc}^{\mu\nu}(x_+) + O(\lambda^2).
\end{aligned} \tag{31}$$

Substituting these expansions into (30) we obtain

$$\begin{aligned}
\mathcal{L}_{\text{quark}} \rightarrow & \bar{q}_s \left( i \mathcal{D}_s + \frac{\not{n}}{2} \bar{n} \cdot A_{sc} \right) q_s + \bar{q}_s S_s \frac{\not{n}}{2} x_{\perp\mu} \bar{n}_\nu g_s G_{sc}^{\mu\nu} S_s^\dagger q_s + O(\lambda^5) \\
& + \left\{ \bar{q}_s S_s \frac{\not{n}}{2} \left( S_s^\dagger [i \bar{n} \cdot D_s + \bar{n} \cdot A_{sc}] S_s - i \bar{n} \cdot D_{sc} \right) \sigma \right. \\
& \left. + \bar{q}_s (i \mathcal{D}_{s\perp} S_s) \left[ (1 + x_\perp \cdot D_{sc}) \theta + \sigma \right] + \text{h.c.} + O(\lambda^{11/2}) \right\},
\end{aligned} \tag{32}$$

where it is now understood that all soft-collinear fields are evaluated at position  $x_+$  (after derivatives have been taken).

The first term in the first line in the above result contains a leading-order interaction of soft quarks with the soft-collinear gluon field  $\bar{n} \cdot A_{sc}$ . This term can be removed by making another redefinition of the soft fields, which is analogous to the decoupling of ultra-soft gluon fields at leading power in SCET<sub>I</sub> [2]. We define

$$q_s(x) = W_{sc}(x_+) q_s^{(0)}(x), \quad A_s^\mu(x) = W_{sc}(x_+) A_s^{(0)\mu}(x) W_{sc}^\dagger(x_+), \tag{33}$$

where

$$W_{sc}(x_+) = \text{P exp} \left( i \int_{-\infty}^0 dt \bar{n} \cdot A_{sc}(x_+ + t\bar{n}) \right) \tag{34}$$

with  $t \sim \lambda^{-1}$ . This object is invariant under soft and collinear gauge transformations, while under a soft-collinear gauge transformation

$$W_{sc}(x_+) \rightarrow U_{sc}(x_+) W_{sc}(x_+). \tag{35}$$

Consequently, the new fields with “(0)” superscripts are invariant under soft-collinear gauge transformations, and there is no longer a soft-collinear background field in their transformation rules. In terms of these fields the first term in (32) reduces to the soft Lagrangian  $\mathcal{L}_s = \bar{q}_s^{(0)} i \not{D}_s^{(0)} q_s^{(0)}$  in (16). The remaining terms yield contributions to the interaction Lagrangian  $\mathcal{L}_{s+sc}^{\text{int}}$  in (16). Using that  $S_s = W_{sc}(x_+) S_s^{(0)} W_{sc}^\dagger(x_+)$  under the transformation (33) we obtain

$$\begin{aligned} \mathcal{L}_{s+sc}^{\text{int}} = & \bar{q}_s^{(0)} S_s^{(0)} \frac{\not{x}_\perp}{2} W_{sc}^\dagger x_{\perp\mu} \bar{n}_\nu g_s G_{sc}^{\mu\nu} W_{sc} S_s^{(0)\dagger} q_s^{(0)} + O(\lambda^5) \\ & + \left\{ \bar{q}_s^{(0)} (i \not{D}_{s\perp}^{(0)} S_s^{(0)}) W_{sc}^\dagger \left[ (1 + x_\perp \cdot D_{sc}) \theta + \sigma \right] + \bar{q}_s^{(0)} \frac{\not{x}_\perp}{2} (i \bar{n} \cdot D_s^{(0)} S_s^{(0)}) W_{sc}^\dagger \sigma \right. \\ & \left. + \text{h.c.} + O(\lambda^{11/2}) \right\}. \end{aligned} \quad (36)$$

This result can be simplified further by introducing the gauge-invariant building blocks [5]

$$\begin{aligned} \mathcal{Q}_s &= S_s^{(0)\dagger} q_s^{(0)} = W_{sc}^\dagger(x_+) S_s^\dagger q_s, \\ \mathcal{A}_s^\mu &= S_s^{(0)\dagger} (i D_s^{(0)\mu} S_s^{(0)}) \\ &= W_{sc}^\dagger(x_+) \left[ S_s^\dagger (i D_s^\mu S_s) + \frac{n^\mu}{2} (S_s^\dagger \bar{n} \cdot A_{sc}(x_+) S_s - \bar{n} \cdot A_{sc}(x_+)) \right] W_{sc}(x_+), \end{aligned} \quad (37)$$

which are invariant under both soft and soft-collinear gauge transformations. In terms of these fields we find the final result

$$\begin{aligned} \mathcal{L}_{s+sc}^{\text{int}} = & \bar{\mathcal{Q}}_s \frac{\not{x}_\perp}{2} W_{sc}^\dagger x_{\perp\mu} \bar{n}_\nu g_s G_{sc}^{\mu\nu} W_{sc} \mathcal{Q}_s + O(\lambda^5) \\ & + \left\{ \bar{\mathcal{Q}}_s \mathcal{A}_{s\perp} W_{sc}^\dagger \left[ (1 + x_\perp \cdot D_{sc}) \theta + \sigma \right] + \bar{\mathcal{Q}}_s \frac{\not{x}_\perp}{2} \bar{n} \cdot \mathcal{A}_s W_{sc}^\dagger \sigma + \text{h.c.} + O(\lambda^{11/2}) \right\}. \end{aligned} \quad (38)$$

Recall that all components of  $x^\mu$  in this Lagrangian scale like  $\lambda^{-1}$ . Soft fields live at position  $x$ , while soft-collinear fields must be evaluated at position  $x_+$ . The small-component field  $\sigma$  may be eliminated using (15). Note that the soft-collinear fields enter this result in combinations such as  $W_{sc}^\dagger \theta$ , which are explicitly gauge invariant.

The measure  $d^4x$  relevant to these interaction terms scales like  $\lambda^{-4}$ . It follows that in terms of the redefined fields the interaction of two soft quarks with a soft-collinear gluon (first line) is a subleading effect, for which we have computed the  $O(\lambda^{1/2})$  contribution to the action. The interaction of a soft quark and soft gluon with a soft-collinear quark is also a subleading effect, for which we have computed the  $O(\lambda^{1/2})$  and  $O(\lambda)$  contributions to the action.

In addition to the quark terms shown above there exist pure-gluon interactions between soft-collinear and soft gluons. It can be readily seen that after the decoupling transformation they are also of subleading order in power counting. The reason is that only the component  $\bar{n} \cdot A_{sc}$  of the soft-collinear field is as large as the corresponding component  $\bar{n} \cdot A_s$  of the soft field. Since the measure  $d^4x$  associated with  $\mathcal{L}_{s+sc}^{\text{int}}$  is the same as that for purely soft interactions, leading-order couplings of soft and soft-collinear gluons can only contain the component  $\bar{n} \cdot A_{sc}$

of the soft-collinear gluon field. However, all such interactions disappear when the soft fields are redefined as in (33), because

$$iD_{s+sc}^\mu = iD_s^\mu + \frac{n^\mu}{2} \bar{n} \cdot A_{sc}(x_+) + \dots = W_{sc}(x_+) iD_s^{(0)\mu} W_{sc}^\dagger(x_+) + \dots, \quad (39)$$

where the dots denote higher-order terms in  $\lambda$ . The remaining subleading interactions between soft and soft-collinear gluons are of lesser phenomenological importance than the terms in (38). Their precise form will not be derived here.

## 5.2 Interactions between soft-collinear fields and collinear fields

The discussion of the sector of the theory involving collinear and soft-collinear fields proceeds in an analogous way. In this case we replace the transformation rules for collinear fields in (21) and (22) by the homogeneous gauge transformations

$$\begin{aligned} \text{collinear: } \quad \bar{n} \cdot A_c &\rightarrow U_c \bar{n} \cdot A_c U_c^\dagger + U_c [i\bar{n} \cdot \partial, U_c^\dagger], & \xi &\rightarrow U_c \xi, \\ A_{c\perp}^\mu &\rightarrow U_c A_{c\perp}^\mu U_c^\dagger + U_c [i\partial_\perp^\mu, U_c^\dagger], \\ n \cdot A_c &\rightarrow U_c n \cdot A_c U_c^\dagger + U_c [in \cdot D_{sc}(x_-), U_c^\dagger], \\ \text{soft-collinear: } \quad A_c^\mu &\rightarrow U_{sc}(x_-) A_c^\mu U_{sc}^\dagger(x_-), & \xi &\rightarrow U_{sc}(x_-) \xi. \end{aligned} \quad (40)$$

Once again the soft-collinear fields transform in the same way as before. The new collinear quark and gluon fields obeying the homogeneous transformation rules are related to the original fields in collinear light-cone gauge  $\bar{n} \cdot A_c = 0$  by the field redefinitions [20]

$$\begin{aligned} \xi|_{\text{CLCG}} &= R_c W_c^\dagger \hat{\xi}, \\ A_{c\perp}^\mu|_{\text{CLCG}} &= R_c W_c^\dagger (i\hat{D}_{c\perp}^\mu W_c) R_c^\dagger, \\ n \cdot A_c|_{\text{CLCG}} &= R_c \left[ W_c^\dagger (in \cdot \hat{D}_c W_c) + W_c^\dagger n \cdot A_{sc}(x_-) W_c - n \cdot A_{sc}(x_-) \right] R_c^\dagger, \end{aligned} \quad (41)$$

where

$$W_c(x) \equiv \text{P exp} \left( i \int_{-\infty}^0 dt \bar{n} \cdot \hat{A}_c(x + t\bar{n}) \right) \quad (42)$$

with  $t \sim 1$  is a collinear Wilson line (expressed in terms of the new gluon field) along the  $\bar{n}$ -direction, and

$$R_c(x) = \text{P exp} \left( i \int_0^1 dt (x - x_-)_\mu A_{sc}^\mu(x_- + t(x - x_-)) \right) \quad (43)$$

is the gauge string of soft-collinear fields from  $x_-$  to  $x$ . The transformation properties of these objects are analogous to those for the Wilson lines  $S_s$  and  $R_s$  introduced in (27) and (28).

To obtain the effective Lagrangian we adopt collinear light-cone gauge and insert the decomposition  $q = \xi + \eta + q_{sc}$  into the Dirac Lagrangian, dropping terms that are forbidden by

momentum conservation. The only difference with respect to the discussion in Section 5.1 is that the small-component field  $\eta$ , which is part of the QCD collinear fermion field, must later be eliminated using its equation of motion. We then expand the terms involving the object  $R_c$  in analogy with (31), and finally redefine the collinear fields in analogy with (33), i.e.

$$\xi(x) = S_{sc}(x_-) \xi^{(0)}(x), \quad \eta(x) = S_{sc}(x_-) \eta^{(0)}(x), \quad A_c^\mu(x) = S_{sc}(x_-) A_c^{(0)\mu}(x) S_{sc}^\dagger(x_-), \quad (44)$$

where

$$S_{sc}(x_-) = \text{P exp} \left( i \int_{-\infty}^0 dt n \cdot A_{sc}(x_- + tn) \right) \quad (45)$$

with  $t \sim \lambda^{-2}$  is defined in a similar way as the object  $W_{sc}$  in (34). In terms of the new fields the Dirac Lagrangian splits up into the collinear Lagrangian  $\mathcal{L}_c$  in (16) and terms that contribute to the interaction Lagrangian  $\mathcal{L}_{c+sc}^{\text{int}}$  in (16). (Since we have separately analyzed the  $(s+sc)$  and  $(c+sc)$  sectors we also obtain another copy of the Lagrangian  $\mathcal{L}_{sc}$ , which must be dropped.) These terms are given by

$$\begin{aligned} \mathcal{L}_{c+sc}^{\text{int}} = & \bar{\xi}^{(0)} W_c^{(0)} \frac{\not{n}}{2} S_{sc}^\dagger x_{\perp\mu} n_\nu g_s G_{sc}^{\mu\nu} S_{sc} W_c^{(0)\dagger} \xi^{(0)} + O(\lambda^5) \\ & + \left\{ \bar{\xi}^{(0)} (i \not{D}_{c\perp}^{(0)} W_c^{(0)}) S_{sc}^\dagger (1 + x_\perp \cdot D_{sc}) \sigma + \bar{\eta}^{(0)} (i \not{D}_{c\perp}^{(0)} W_c^{(0)}) S_{sc}^\dagger \theta \right. \\ & \left. + \bar{\xi}^{(0)} \frac{\not{n}}{2} (i n \cdot D_c^{(0)} W_c^{(0)}) S_{sc}^\dagger \theta + \text{h.c.} + O(\lambda^{11/2}) \right\}, \end{aligned} \quad (46)$$

where all soft-collinear fields are now evaluated at  $x_-$ . This result can be simplified further by introducing the gauge-invariant building blocks [5]

$$\begin{aligned} \mathcal{X} &= W_c^{(0)\dagger} \xi^{(0)} = S_{sc}^\dagger(x_-) W_c^\dagger \xi, \\ \mathcal{A}_c^\mu &= W_c^{(0)\dagger} (i D_c^{(0)\mu} W_c^{(0)}) \\ &= S_{sc}^\dagger(x_-) \left[ W_c^\dagger (i D_c^\mu W_c) + \frac{\bar{n}^\mu}{2} (W_c^\dagger n \cdot A_{sc}(x_-) W_c - n \cdot A_{sc}(x_-)) \right] S_{sc}(x_-), \end{aligned} \quad (47)$$

which are invariant under both collinear and soft-collinear gauge transformations. Expressing the result in terms of these fields, and eliminating the field  $\eta^{(0)}$  using its leading-order equation of motion

$$\eta^{(0)} = -\frac{\not{n}}{2} \frac{1}{i \bar{n} \cdot D_c^{(0)}} i \not{D}_{c\perp}^{(0)} \xi^{(0)} + \dots, \quad (48)$$

we obtain the final result

$$\begin{aligned} \mathcal{L}_{c+sc}^{\text{int}} = & \bar{\mathcal{X}} \frac{\not{n}}{2} S_{sc}^\dagger x_{\perp\mu} n_\nu g_s G_{sc}^{\mu\nu} S_{sc} \mathcal{X} + O(\lambda^5) \\ & + \left\{ \bar{\mathcal{X}} \mathcal{A}_{c\perp} S_{sc}^\dagger (1 + x_\perp \cdot D_{sc}) \sigma - \bar{\mathcal{X}} i \not{D}_{c\perp} \frac{\not{n}}{2} \frac{1}{i \bar{n} \cdot \partial} \mathcal{A}_{c\perp} S_{sc}^\dagger \theta + \bar{\mathcal{X}} \frac{\not{n}}{2} n \cdot \mathcal{A}_c S_{sc}^\dagger \theta \right. \\ & \left. + \text{h.c.} + O(\lambda^{11/2}) \right\}, \end{aligned} \quad (49)$$



where  $i\mathcal{D}_c^\mu = i\partial^\mu + \mathcal{A}_c^\mu$ . Recall that the components of  $x^\mu$  in this Lagrangian scale like  $(1, \lambda^{-2}, \lambda^{-1})$ . Soft-collinear fields live at position  $x_-$ , while collinear fields live at position  $x$ . The small-component field  $\sigma$  may be eliminated using (15). Once again, the soft-collinear fields enter the result in combinations such as  $S_{sc}^\dagger \theta$ , which are explicitly gauge invariant.

In complete analogy to the  $(s + sc)$  sector it follows that in terms of the redefined fields the interaction of two collinear quarks with a soft-collinear gluon (first line) is a subleading effect, for which we have computed the  $O(\lambda^{1/2})$  contribution to the action. The interaction of a collinear quark and collinear gluon with a soft-collinear quark is also a subleading effect, for which we have computed the  $O(\lambda^{1/2})$  and  $O(\lambda)$  contributions to the action. In addition to the quark terms shown above there exist pure-gluon interactions between soft-collinear and collinear gluons, which are again of subleading order in power counting. Their precise form will not be derived here.

## 6 Induced soft-collinear interactions

Above we have shown that there are no leading-order interactions between soft, collinear, and soft-collinear fields. The three interaction terms in (16) vanish at leading order in  $\lambda$ . This property of SCET<sub>II</sub> is crucial to the idea of soft-collinear factorization, which is at the heart of QCD factorization theorems. In order to preserve a transparent power counting it is then convenient to define hadron states in the effective theory as eigenstates of one of the two leading-order Lagrangians  $\mathcal{L}_s$  and  $\mathcal{L}_c$ . For instance, a ‘‘SCET pion’’ would be a bound state of only collinear fields, and a ‘‘SCET  $B$  meson’’ would be a bound state of only soft fields.

Since by definition these states do not contain any soft-collinear modes there is no need to include source terms for soft-collinear fields in the functional integral. ‘‘Integrating out’’ the soft-collinear fields from the path integral then gives rise to induced, highly non-local interactions between soft and collinear fields. The corresponding term in the action is of the form

$$S_{s+c}^{\text{induced}} = i \int d^4x d^4y \text{T} \left\{ \mathcal{L}_{c+sc}^{\text{int}}(x) \mathcal{L}_{s+sc}^{\text{int}}(y) e^{i \int d^4z \mathcal{L}_{sc}(z)} \right\} \quad (50)$$

with all soft-collinear fields contracted. This result can be expressed in terms of exact, gauge-invariant two-particle correlation functions of soft-collinear fields.

Consider first the effective interaction between two soft and two collinear quarks arising from the exchange of a soft-collinear gluon. The result can be expressed in terms of the correlation function

$$\begin{aligned} \langle \Omega | \text{T} \left\{ (S_{sc}^\dagger n_\nu g_s G_{sc}^{\mu\nu} S_{sc})_a(x_-) (W_{sc}^\dagger \bar{n}_\beta g_s G_{sc}^{\alpha\beta} W_{sc})_b(y_+) e^{i \int d^4z \mathcal{L}_{sc}(z)} \right\} | \Omega \rangle \\ = \delta_{ab} g_\perp^{\mu\alpha} \Delta_G(x_- \cdot y_+) + \dots, \end{aligned} \quad (51)$$

where  $a, b$  are color indices, and the dots represent terms that vanish when contracted with vectors in the transverse plane. SCET power counting implies that  $\Delta_G(x_- \cdot y_+) \sim \lambda^6$ . At tree level we find that  $\Delta_G(x_- \cdot y_+) = -ig_s^2 \delta^{(4)}(z) = -2ig_s^2 \delta(x \cdot \bar{n}) \delta(y \cdot n) \delta^{(2)}(0)$ , where  $z = x_- - y_+$ .<sup>2</sup>

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<sup>2</sup>In  $d = 4 - 2\epsilon$  dimensions,  $\Delta_G = g_s^2 \pi^{-2+\epsilon} \epsilon \Gamma(2 - \epsilon) (2x_- \cdot y_+ + i0)^{-2+\epsilon}$ .

The resulting contribution to the induced interaction in (50) is given by

$$S_{s+c}^{\text{induced},(1)} = i \int d^4x d^4y x_\perp \cdot y_\perp \Delta_G(x_- \cdot y_+) (\bar{\mathcal{X}} \frac{\not{y}}{2} t_a \mathcal{X})(x) (\bar{\mathcal{Q}}_s \frac{\not{y}}{2} t_a \mathcal{Q}_s)(y), \quad (52)$$

which is of  $O(\lambda)$  in power counting, as indicated by the superscript “(1)”.

Next we discuss the induced interactions obtained from the exchange of a soft-collinear quark, as illustrated by the second diagram in Figure 1. The various correlation functions arising in the product of the relevant terms in (38) and (49) can all be related to the covariant expansion of the correlator

$$\begin{aligned} & \langle \Omega | T \left\{ \left[ S_{sc}^\dagger(x_-) (R_c^\dagger q_{sc})(x) \right]_i \left[ (\bar{q}_{sc} R_s)(y) W_{sc}(y_+) \right]_j e^{i \int d^4z \mathcal{L}_{sc}(z)} \right\} | \Omega \rangle \\ &= i \delta_{ij} \left[ (\not{z}_+ + \not{z}_-) \Delta_1(z_+ \cdot z_-, z_\perp^2) + \not{z}_\perp \frac{\not{y} \not{y}}{4} \Delta_2(z_+ \cdot z_-, z_\perp^2) + \not{z}_\perp \frac{\not{y} \not{y}}{4} \Delta_3(z_+ \cdot z_-, z_\perp^2) \right] \end{aligned} \quad (53)$$

about the points  $x = x_-$  and  $y = y_+$ . Here  $i, j$  are color indices. The form of the right-hand side of this equation holds to any (finite) order in perturbation theory. Translational invariance implies that the result is a function of  $z = x - y$ . Note, however, that the presence of the light-cone vectors  $n$  and  $\bar{n}$  in the gauge strings  $S_{sc}$  and  $W_{sc}$  breaks the rotational symmetry between the longitudinal and transverse components of  $z$ . Nevertheless, the expression is symmetric under the simultaneous interchange of  $n \leftrightarrow \bar{n}$  and  $x \leftrightarrow y$  followed by hermitean conjugation. Expanding the result (53) to first order in transverse displacements we find six non-zero correlators, four of which are relevant to our analysis. They are

$$\begin{aligned} & \langle \Omega | T \left\{ (S_{sc}^\dagger \theta)_i(x_-) (\bar{\theta} W_{sc})_j(y_+) e^{i \int d^4z \mathcal{L}_{sc}(z)} \right\} | \Omega \rangle = i \delta_{ij} \not{z}_- \Delta_1(x_- \cdot y_+), \\ & \langle \Omega | T \left\{ (S_{sc}^\dagger \sigma)_i(x_-) (\bar{\sigma} W_{sc})_j(y_+) e^{i \int d^4z \mathcal{L}_{sc}(z)} \right\} | \Omega \rangle = -i \delta_{ij} \not{y}_+ \Delta_1(x_- \cdot y_+), \\ & \langle \Omega | T \left\{ (S_{sc}^\dagger \sigma)_i(x_-) (\bar{\theta} (-i \overleftarrow{D}_{sc\perp}^\mu) W_{sc})_j(y_+) e^{i \int d^4z \mathcal{L}_{sc}(z)} \right\} | \Omega \rangle = -\delta_{ij} \frac{\not{y} \not{y}}{4} \gamma_\perp^\mu \Delta_2(x_- \cdot y_+), \\ & \langle \Omega | T \left\{ (S_{sc}^\dagger i D_{sc\perp}^\mu \sigma)_i(x_-) (\bar{\theta} W_{sc})_j(y_+) e^{i \int d^4z \mathcal{L}_{sc}(z)} \right\} | \Omega \rangle = -\delta_{ij} \frac{\not{y} \not{y}}{4} \gamma_\perp^\mu \Delta_2(x_- \cdot y_+), \end{aligned} \quad (54)$$

where  $\Delta_n(x_- \cdot y_+) \equiv \Delta_n(-x_- \cdot y_+, 0)$ . “Mixed” correlators without an extra transverse derivative such as  $\langle (S_{sc}^\dagger \sigma)(x_-) (\bar{\theta} W_{sc})(y_+) \rangle$  vanish due to the fact that, according to (14) and (15), they involve an odd number of  $\gamma_\perp$  matrices but no transverse Lorentz index. SCET power counting implies that the functions  $\Delta_n(x_- \cdot y_+) \sim \lambda^6$ . In fact, at tree level we obtain  $\Delta_n(x_- \cdot y_+) = 1/[8\pi^2(x_- \cdot y_+)^2]$  for  $n = 1, 2, 3$ .<sup>3</sup> (Recall that  $x_- \sim \lambda^{-2}$  and  $y_+ \sim \lambda^{-1}$ .) It is

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<sup>3</sup>In  $d = 4 - 2\epsilon$  dimensions, the expression is  $\frac{1}{2} \pi^{-2+\epsilon} \Gamma(2-\epsilon) (2x_- \cdot y_+ + i0)^{-2+\epsilon}$ .

now straightforward to evaluate the corresponding terms in the action (50). We find

$$\begin{aligned}
S_{s+c}^{\text{induced},(3/2)} = & \int d^4x d^4y \Delta_1(x_- \cdot y_+) \left\{ \bar{\mathcal{X}} \mathcal{A}_{c\perp} \not{y}_+ \left( \frac{\not{y}}{2} \bar{n} \cdot \mathcal{A}_s + \mathcal{A}_{s\perp} \right) \mathcal{Q}_s \right. \\
& \left. - \bar{\mathcal{X}} \left( \frac{\not{y}}{2} n \cdot \mathcal{A}_c - i \not{\mathcal{D}}_{c\perp} \frac{1}{i\bar{n} \cdot \partial} \frac{\not{y}}{2} \mathcal{A}_{c\perp} \right) \not{x}_- \mathcal{A}_{s\perp} \mathcal{Q}_s \right\} \\
& - \int d^4x d^4y \Delta_2(x_- \cdot y_+) \bar{\mathcal{X}} \mathcal{A}_{c\perp} (\not{x}_\perp - \not{y}_\perp) \mathcal{A}_{s\perp} \mathcal{Q}_s + \text{h.c.} , \tag{55}
\end{aligned}$$

where it is understood that all collinear fields live at  $x$ , while all soft fields live at  $y$ . Power counting shows that this induced long-range soft-collinear interaction is of  $O(\lambda^{3/2})$ , as indicated by the superscript. We stress that the superficially leading term of  $O(\lambda)$  vanishes due to rotational invariance in the transverse plane.

At the same order in power counting there appear terms in the interaction Lagrangian  $\mathcal{L}_{s+c}^{\text{int}}$  in (16) from integrating out hard-collinear modes. At tree level, hard-collinear gluon exchange induces the interaction

$$\mathcal{L}_{s+c}^{\text{int},(1)} = -4\pi\alpha_s \left[ \bar{\mathcal{X}}(x_+ + x_\perp) \frac{\gamma_\mu t_a}{i\bar{n} \cdot \partial} \mathcal{Q}_s(x_- + x_\perp) \right] \left[ \bar{\mathcal{Q}}_s(x_- + x_\perp) \frac{\gamma^\mu t_a}{i\bar{n} \cdot \partial} \mathcal{X}(x_+ + x_\perp) \right] , \tag{56}$$

which should be understood as a matching contribution to the effective Lagrangian at the hard-collinear scale  $\mu^2 \sim E\Lambda$ . This operator (in Fierz-transformed form) has been derived previously in a discussion of color-suppressed hadronic  $B$  decays in [21]. Power counting shows that it scales like  $\lambda^4$ , which when combined with the measure  $d^4x \sim \lambda^{-3}$  indeed gives rise to an  $O(\lambda)$  interaction term in the action. Note that in terms containing both soft and collinear fields, the soft fields must be multipole expanded about  $x_+ = 0$ , while the collinear fields must be expanded about  $x_- = 0$ . This point was not emphasized in [5]. The operator obtained from the exchange of a hard-collinear quark shown in Figure 1 has the form

$$\begin{aligned}
\mathcal{L}_{s+c}^{\text{int},(3/2)} = & -\bar{\mathcal{X}} \frac{1}{i\bar{n} \cdot \partial} \mathcal{A}_{s\perp} \frac{\not{y}}{2} \mathcal{A}_{c\perp} \mathcal{Q}_s - \bar{\mathcal{X}} \frac{1}{i\bar{n} \cdot \partial} \mathcal{A}_{s\perp} (i\not{\partial}_\perp + \mathcal{A}_{s\perp} + \mathcal{A}_{c\perp}) \mathcal{A}_{c\perp} \frac{1}{i\bar{n} \cdot \partial} \mathcal{Q}_s \\
& - \bar{\mathcal{X}} (i\not{\partial}_\perp + \mathcal{A}_{c\perp}) \frac{1}{i\bar{n} \cdot \partial} \mathcal{A}_{s\perp} \mathcal{A}_{c\perp} \frac{1}{i\bar{n} \cdot \partial} \mathcal{Q}_s + \text{h.c.} , \tag{57}
\end{aligned}$$

where, as in (56), all collinear fields are to be evaluated at the point  $x_+ + x_\perp$  and the soft fields at  $x_- + x_\perp$ .

Translated to momentum space, this particular position dependence of the fields enforces that the minus component of the total collinear momentum flowing into the vertex exits again through collinear lines. An analogous statement holds for the plus component of the soft momenta. The momentum-conservation  $\delta$ -functions associated with the above vertices therefore reads  $\delta(\bar{n} \cdot P) \delta(n \cdot L) \delta^{(2)}(P_\perp + L_\perp)$ , where  $P$  is the sum of all collinear and  $L$  the sum of all soft momenta (ingoing minus outgoing) connected to the vertex. As classical scattering processes, these interactions are rather exceptional: the operator (56), for example, contributes only to the forward scattering of a collinear quark and a soft quark. When inserted

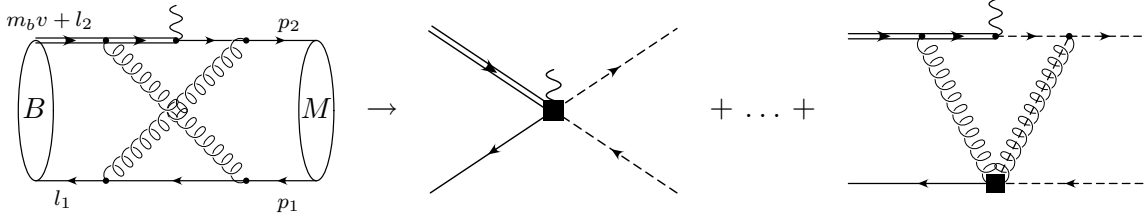


Figure 5: A QCD diagram contributing to the decay of a  $B$  meson to an energetic light meson  $M$ , and its representation in the effective theory. Solid lines carry soft, dashed lines collinear momentum. External soft momenta are incoming and collinear ones outgoing. The last diagram, which involves the interaction (57), vanishes.

into loop diagrams, such exceptional momentum configurations in general do not give rise to non-zero contributions. To see this, let us analyze the one-loop QCD diagram shown in Figure 5 using the strategy of regions. One finds that the graph receives a contribution from the region where all propagators are hard or hard-collinear. In the effective theory this contribution is represented by the first diagram on the right-hand side. There are also contributions from regions where two propagators of the loop are soft (or collinear) and all others hard or hard-collinear. These correspond to effective-theory diagrams (not shown) where a collinear or soft gluon is emitted from the weak vertex and absorbed by one of the quarks. However, no contribution arises that would correspond to the last diagram in Figure 5, which involves the interaction (57) denoted by the black square. First, it is impossible to assign a loop momentum and external momenta in the full-theory diagram such that the exceptional momentum configuration corresponding to the last graph (where the gluon connected to the heavy quark is soft and that connected to the outgoing light quark is collinear) would be enforced. This configuration would require the loop momentum to scale like  $k^\mu \sim (\lambda^2, \lambda, \lambda)$ , in which case the denominators of the QCD loop integral can be simplified as

$$\begin{aligned} & \frac{1}{2l_{1+} \cdot (k_- + l_{1-}) + (k_\perp + l_{1\perp})^2 + i0} \frac{1}{v \cdot (k_- + k_\perp + l) + i0} \\ & \times \frac{1}{2p_{1-} \cdot (k_+ + p_{1+}) + (k_\perp + p_{1\perp})^2 + i0} \frac{1}{2p_- \cdot (k_+ + p_+) + (k_\perp + p_\perp)^2 + i0}, \end{aligned} \quad (58)$$

where  $l = l_1 + l_2$  and  $p = p_1 + p_2$ . These denominator structures precisely correspond to the multipole-expanded form of the operator (57). Note that the plus component of the loop momentum does not propagate into soft lines, while its minus component does not propagate into collinear lines. As a result, the integration over  $k_-$  vanishes if the components  $n \cdot l_i$  of the external soft momenta all have the same sign, because then the contour can be deformed away from the poles. Similarly, the integration over  $k_+$  vanishes if the components  $\bar{n} \cdot p_i$  of the external collinear momenta all have the same sign. The same conclusion is reached upon analyzing a general loop diagram involving (56) or (57): non-zero contributions from these interactions can only occur in processes which involve soft and collinear particles in both

the initial and final state, and these interactions therefore do not contribute in decays of  $B$  mesons. This is a consequence of the Coleman–Norton theorem [12], which states that on-shell singularities in Feynman diagrams correspond to classically allowed scattering processes. The above conclusion also applies to the analysis in [21]. The non-perturbative soft-collinear rescattering mechanism introduced there should not be described in terms of (56), but rather through the long-distance messenger exchange interaction (52).

As a final note, let us mention that the same arguments ensure that there are no induced self-interactions of the type (50) among only soft or only collinear fields, i.e., the effective Lagrangians  $\mathcal{L}_s$  and  $\mathcal{L}_c$  are not altered by integrating out the soft-collinear modes. The corresponding correlation functions  $\langle (S_{sc}^\dagger \theta)(x_-) (\bar{\theta} S_{sc})(y_-) \rangle$  etc. vanish, since the contour can be closed avoiding the poles in the complex plane. This ensures that soft-collinear fields can only be exchanged in graphs involving both soft and collinear fields.

## 7 Conclusions

We have argued that the version of soft-collinear effective theory appropriate for the discussion of exclusive  $B$  decays such as  $B \rightarrow \pi l \nu$ ,  $B \rightarrow K^* \gamma$ , and  $B \rightarrow \pi \pi$  should include a new mode in addition to soft and collinear fields, which can interact with both soft and collinear particles without taking them far off-shell. The relevance of these messenger fields has been demonstrated using the example of a triangle diagram with one soft and one collinear external momentum. We have then developed the formalism incorporating the corresponding quark and gluon fields into the soft-collinear effective Lagrangian.

In the strong-interaction sector of the effective theory the leading-order interactions of soft-collinear fields with soft or collinear fields can be removed using field redefinitions. The remaining interactions are power suppressed. We have explicitly worked out the leading terms involving quark fields. We stress that this decoupling in the QCD sector of the theory does not necessarily imply that soft-collinear modes can be ignored at leading order in power counting. This would only be true in cases where the decoupling transformation leaves external operators such as weak-interaction currents invariant. When the messenger fields are “integrated out” in the functional integral they give rise to highly non-local induced interactions between soft and collinear fields, which appear at the same order in power counting as short-distance interactions among these fields induced by the exchange of hard-collinear modes. However, only the long-distance interactions induced by soft-collinear exchange graphs are kinematically allowed in  $B$  decays, where no collinear particles are present in the initial state.

A puzzling aspect of our analysis is the finding of a relevant momentum region corresponding to virtualities that are parametrically below the QCD scale  $\Lambda_{\text{QCD}}$ . However, our discussion in Section 3 illustrates that the same phenomenon arises in the case of the Sudakov form factor, if the external massless particles have off-shell momenta scaling like  $p_1^\mu \sim (\Lambda^2/E, E, \Lambda)$  and  $p_2^\mu \sim (E, \Lambda^2/E, \Lambda)$  with  $\Lambda \sim \Lambda_{\text{QCD}}$ . Then perturbatively the ultra-soft region of loop momenta  $k^\mu \sim \Lambda^2/E$  gives rise to leading-order contributions and acts as a “messenger” between the two collinear particles, just like the soft-collinear modes connecting soft and collinear particles in our case.

The fact that modes with  $p^2 \ll \Lambda^2$  represent very long-range fluctuations compared to the scale of hadronic systems is not necessarily an argument against their existence. It is to some extent a consequence of dimensional regularization and analyticity that the relevant momentum region is  $k^\mu \sim E(\lambda^2, \lambda, \lambda^{3/2})$  and not  $k^\mu \sim E(\lambda^2, \lambda, \lambda)$ , in which case the virtuality would be of order  $\Lambda^2$ . Ultimately, it is not the virtuality that is important but the fact that the plus and minus components of soft-collinear momenta are commensurate with certain components of collinear or soft momenta. As an analogy, let us recall the case of the hydrogen Lamb shift, in which “ultra-soft” modes with time and space components of momentum scaling like  $(mv^2, mv^2)$  are important. These modes have virtuality  $p^2 \ll (mv)^2$ , where  $mv$  is the typical momentum of the hydrogen atom. Also in this case it is not the off-shellness that matters, but rather the fact that one of the components (the energy) is comparable to that of the bound-state system.

Finally, let us mention that the introduction of a mass term  $m \sim \lambda$  in the propagator labeled with  $k$  in Figure 2 would change the analysis of regions. In this case there would be no contribution from the soft-collinear region, since the loop momentum  $k^2 \sim \lambda^3$  in the soft-collinear propagator  $1/(k^2 - m^2)$  could be Taylor-expanded and no pinch singularities would occur. However, this “simplification” comes at a price: after the introduction of the mass term, the loop integrals occurring in the soft and collinear region are no longer regulated dimensionally. In order to obtain the expansion of the massive integral using the strategy of regions, additional analytic regulators have to be introduced on the soft and collinear propagators [18]. The procedure is straightforward as long as one is only interested in the expansion of a given loop integral, but would complicate the construction of the effective theory since the analytic regulators destroy longitudinal boost and gauge invariance. We believe it is fair to say that to date it is not known how to consistently incorporate masses of  $O(\Lambda)$  into soft-collinear effective theory.

A related question one might worry about is whether the scaling laws for the propagator functions  $\Delta_n(x_- \cdot y_+)$  describing the soft-collinear exchange, which follow from the power-counting rules of the effective theory, could be invalidated by some non-perturbative effects not seen at the level of Feynman diagrams, such as a generation of mass terms in exact propagators. In our opinion such effects would not only upset power counting but threaten the usefulness of the effective theory as a whole. For now we take the conservative point of view that in order to be useful the effective theory should at least reproduce correctly the analytic properties of perturbation theory. As we have shown, the inclusion of soft-collinear modes is then an indispensable part in the construction of the effective Lagrangian.

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